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Generalized boundary conditions of the heat and mass transfer

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Abstract—The genesis of the statement of classical and generalized boundary conditions is considered in relation to parabolic and hyperbolic heat conduction equations. Differential-type boundary conditions of any orders, that involve all earlier known boundary conditions, are generalized for hyperbolic equations; the possibility for the formulation of boundary conditions in integral or integrodifferential forms of hyperbolic equations in heat and mass transfer theory, electrodynamics and thermomechanics of media with memory or with relaxation processes is shown.

INTRODUCTION

THE BOUNDARY conditions in transport theory, just as in the general theory of partial differential equations, are of fundamental importance since they determine the explicit form of the solution. This is because the general solutions of such equations are determined accurately for arbitrary functions. The specific form of an arbitrary function can be found only from additional conditions proceeding from the prescribed physical processes on the boundaries of the region considered, from the mathematical requirement that the solution should be finite or from its prescribed behaviour at singular points, lines, discontinuity surfaces or at far removed points. The statement of boundary conditions frequently determines the mathematical correctness of a boundary-value problem according to Hadamard–Tikhonov. The boundary conditions on interfaces are usually formulated in physical problems with account for the basic conservation laws in integral differential or other forms. In the classical heat conduction theory, based on the Fourier law and the respective parabolic heat conduction equation

$$\rho c_{\mathrm{p}} \frac{\partial T}{\partial t} = \operatorname{div} \lambda \operatorname{grad} T - \kappa T + F \qquad (1)$$

with the initial condition at

$$t = 0 \quad T(t, x, y, z) = T_0(0, x, y, z) \qquad (2)$$

four boundary conditions were usually investigated [1, 2]:

(i) The first kind boundary condition is that when the following unknown function is prescribed on the bounding surface:

$$T|_{\Gamma} = \varphi. \qquad (3)$$

(ii) The second kind boundary condition consists of the assignment of the unknown function derivative along the normal to the surface at the interface

$$\lambda \left.\frac{\partial T}{\partial n}\right|_{\Gamma} \equiv q|_{\Gamma} = \Psi. \qquad (4)$$

(iii) The third kind boundary condition consists in the assignment of a linear combination of the unknown function with its normal derivative at the interface

$$|\alpha T + \beta q|_{\Gamma} = \zeta. \qquad (5)$$

(iv) The fourth kind boundary condition is that when the unknown functions (temperatures) and heat fluxes are equal at the boundary, provided there is a perfect contact between two media and no heat release at the interface

$$T_1|_{\Gamma+0} = T_2|_{\Gamma-0}, \quad q_1|_{\Gamma+0} = q_2|_{\Gamma-0}. \qquad (6)$$

Different modifications and means of these boundary conditions for other types of equations are known. For example, for the Laplace elliptic equation in the theory of complex variable functions and for the Poisson elliptic equation in the theory of the potential the first condition is called, following Riemann [3], the condition of Dirichlet; the second boundary condition is called the Neumann condition; the third boundary condition is called a mixed one, or the condition with an oblique derivative. The fourth boundary condition is called the condition of conjugation. A particular separate case of these are periodic boundary conditions.

It should be noted that boundary conditions (i)–(iv) for elliptic, parabolic and hyperbolic equations were encountered as long ago as the classical works of Taylor, Bernoulli, D'Alembert, Laplace, Fourier, Poisson, Ostrogradsky and others. Therefore, for the sake of historical justice, it would be better to call these boundary conditions after the scientists who first formulated them. Strictly speaking, the Dirichlet problem is the name given by Riemann in 1857, in honour of Lejeune-Dirichlet [3], to the boundary-

NOMENCLATURE

a, b boundaries of the segment $a \leqslant z \leqslant b$
$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ prescribed functions or constants
c concentration
C linear capacity
C_0 concentration capacity at $z = a$ and $z = b$
c_p heat capacity
D diffusion coefficient
E prescribed external voltage
F function of heat sources (sinks)
G linear conductivity
i current
i_0 prescribed initial function
j_m mass flux
k coefficient
$K(t-\tau)$ kernel of equation
L linear inductance
L_0 concentrated inductance at $z = a$ and $z = b$
n concentration
q heat flux
q_0 initial function of heat flux
Q density of mass sources (sinks)
R linear resistance
R_0 concentrated resistance at $x = a$ and $z = b$
t time
T temperature
T_0 initial function of temperature
u voltage
u_0 prescribed initial function
z coordinate.

Greek symbols
α, β coefficients
α_{ij}, β_{ij} coefficients
$\alpha(0)$ instantaneous volumetric heat capacity
$\alpha'(\tau)$ internal energy relaxation function
Γ boundary
κ coefficient
$\lambda(0)$ instantaneous thermal conductivity coefficient
ξ prescribed time function at the boundary
ρ density
τ_m mass flux relaxation time
τ_q heat flux relaxation time
φ prescribed time function at the boundary
Φ_0 prescribed function
ψ prescribed time function at the boundary
Ψ_i prescribed function.

value problem on the search of the harmonic function by the prescribed values of the function at all the points of the boundary. Riemann also pointed to the possibility for the formulation of other boundary conditions. The first kind boundary conditions were encountered in the works of D'Alembert (1747), Euler (1748) and Bernoulli (1747) in their famous debate about the representation of the function (see [3]), as well as in the works of other scientists for hyperbolic equations in the theory of oscillations of jets, rods and membranes; in the heat conduction theory they were formulated by Fourier (1822) [6], Ostrogradsky (1827) [6], Poisson (1835) [7] and others; in the electromagnetism theory they were considered by Green (1828) [8] and Gauss (1840), among others. Different extensions of boundary conditions in the theory of analytical functions were made by Hilbert, Poincaré, Volterra, Vekua, Gakhov and others [10].

The second kind boundary conditions were considered in the heat conduction theory in the works of Fourier (1822) [6], Ostrogradsky (1826) [7], Poisson [11], Lame [12]; in fluid dynamics—in the works of Ostrogradsky [7] and others; in electrodynamics—in the works of Kirchhoff (1848) [13, 14] and Neumann [15].

The third kind boundary conditions in the heat conduction theory were first considered by Fourier. Here they correspond to the Newton–Richman heat transfer law; while in electrodynamics they correspond to the impedance boundary condition of Shchukin and Leontovich. The third kind boundary conditions were also encountered in the works of Poisson, Ostrogradsky and Lame [12].

Also encountered in the works of Fourier (1822) [6], Ostrogradsky (1828) [7] and Poincaré [16] are the following boundary conditions

$$m\frac{\partial T}{\partial x}+n\frac{\partial T}{\partial y}+i\frac{\partial T}{\partial z}-hT=0,$$

which are also called the conditions with an oblique derivative, or the Poincaré problem.

In the heat conduction theory the third kind boundary conditions were challenged by Luikov [17] in connection with the development of conjugated boundary-value problems.

The fourth kind boundary conditions were encountered in the heat conduction theory in the works of Fourier (1822) [6] (in his problem of a ring), Kirchhoff [18] and Riemann [19], and in the theory of vibrations of composite strings—in the works of Bernoulli [5], and others.

For a second-order hyperbolic equation the assignment of two initial conditions is required: of the function proper, and of its derivative at $t = 0$

$$T = T_0(0,x,y,z), \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = T'_0(0,x,y,z). \quad (7)$$

The initial conditions of the form of (2) for a parabolic equation and of the form of (7) for a hyperbolic equation are also called the Cauchy conditions, although the initial conditions for a second-order hyperbolic equation—the equation for the oscillations of strings—was first formulated by Euler in 1748; in heat conduction theory they were first encountered in the works of Fourier [6]. It should also be noted here that the parabolic equation of type (1) was also first encountered in the works of Euler [2] as were the Laplace elliptic-type equations and different types of hyperbolic and mixed equations. The modern classification of the types of equations was made in the works of Laplace, Monge and DuBois-Reymond [21]. The latter was the first to classify quasi-linear, second-order, partial differential equations into elliptic, hyperbolic and parabolic types. Mixed types of differential equations were set apart later in the works of Tricomi [22], Frankl [23], Lavrentiev, Bitsadze [24], and others. This classification should always be borne in mind when considering and stating boundary conditions.

Thus, for hyperbolic-type equations, apart from boundary conditions (i)–(iv), the statement of other conditions is also possible. When the conditions are prescribed on the non-characteristic curve, this corresponds to the Cauchy problem; the assignment of conditions on the characteristics corresponds to the Darbu problem; when part of the conditions is prescribed on characteristic curves and part on non-characteristic curves, this corresponds to different mixed problems. For mixed-type equations the statement of different boundary conditions is possible—such as the problems of Tricomi, Frankl, etc.

GENERALIZED DIFFERENTIAL-TYPE BOUNDARY CONDITIONS

Boundary conditions (i)–(iv) are not the only types of boundary conditions that are prescribed. It is pertinent here to mention the well-known mathematical problems of Gilbert who, in his work "The general problem of boundary conditions", advanced, as one of the unresolved problems, the 20th one which is

> ... closely connected with those above and is the problem on the existence of solutions to partial differential equations with prescribed boundary conditions. Ingenious methods of Schwarz, Neumann, Poincaré have in the main led to the solution of this problem for the case of the differential equation of the potential, but, nevertheless, these methods cannot be directly extended to the case when the values of derivatives or relationships between these values and those of the unknown function are prescribed on the boundaries ... [25].

This suggests the idea that different generalizations of boundary conditions are possible and have been undertaken in different areas of mathematical physics.

Some generalizations of boundary conditions pertaining to the heat and mass transfer theory will now be considered.

Even a formal generalization of boundary conditions (i)–(iv) leads to the statement of generalized boundary conditions—called by the present author the fifth boundary-value problem (v) [26–29]. It consists in the solution of equation (1) with initial condition (2) under the following boundary conditions:

$$\begin{aligned} R_1\left(T,\frac{\partial T}{\partial z}\right) &= \left|a_1 T + a_2\frac{\partial T}{\partial z}\right|_{z=a} + \left|a_3 T + a_4\frac{\partial T}{\partial z}\right|_{z=b} \\ R_2\left(T,\frac{\partial T}{\partial z}\right) &= \left|b_1 T + b_2\frac{\partial T}{\partial z}\right|_{z=a} + \left|b_3 T + b_4\frac{\partial T}{\partial z}\right|_{z=b}. \end{aligned} \quad (8)$$

The fifth kind boundary-value problem (8) involves, as a special case, boundary conditions (i)–(iv) that result from relations (8) by means of an appropriate choice of the coefficients a_i and b_i. It turned out that boundary conditions (8) are of importance in their own right and were investigated in detail by Steklov as early as in 1923 [30]. He showed that depending on the differences composed of the coefficients

$$a_i b_k - a_k b_i \quad (i,k = 1,2,3,4)$$

two cases are possible: (1) when the difference is not equal to zero; or (2) one of the differences is equal to zero while any other is not zero. In the first case, conditions (8) are reduced to the canonical form

$$\begin{aligned} \frac{\partial T}{\partial z}(t,b) &= \alpha T(t,a) + \beta T(t,b) \\ \frac{\partial T}{\partial z}(t,a) &= \gamma T(t,a) + \delta T(t,b) \end{aligned} \quad (9)$$

and in the second case to

$$\begin{aligned} T(t,b) &= \rho T(t,a) \\ \frac{\partial T}{\partial z}(t,b) &= \sigma\frac{\partial T}{\partial z}(t,a) + \tau T(t,a) \end{aligned} \quad (10)$$

where α, β, γ, δ, ρ, σ, τ are some constants.

Vladimirov [30] indicated still other independent cases resulting from (8). For example, the third case

$$\begin{aligned} T(t,b) &= \rho T(t,a) \\ b_1 T(t,a) &= b_2\frac{\partial T}{\partial z}(t,a) \end{aligned} \quad (11)$$

at $b_1^2 + b_2^2 \neq 0$ and $\rho b_2 = 0$ and the cases that result from relations (11) on replacing a by b. Steklov also mentioned the physical application of the resulting

> ... conditions (9) which comprise the problems on the cooling of non-closed solid bodies of linear dimensions (a straight rod and a rod bent into a non-closed curve), while the second conditions (10) correspond to the same problems of closed solid bodies of linear dimensions (continuous ring, rod bent into a closed curve). These two

classes of problems, different in physical aspects, are somewhat peculiar from the viewpoint of a pure analysis and therefore merit particular consideration.

The latter two cases of the form of equations (11) are called exceptional and are considered separately by Steklov in ref. [30, Chap. 6].

Thus, it has turned out that, apart from formal significance, boundary conditions (8) have also the physical meaning and mathematical independence as a special class of boundary conditions. This allowed the author to refer to these conditions as Steklov's boundary-value problem [28, 29].

Boundary conditions (8) contain a linear combination of unknown functions and their first spatial derivatives.

In the heat conduction theory and mathematical physics [2, 31] boundary conditions were considered that contain a linear form of the unknown function and its first space and time derivatives, i.e. the boundary conditions of the form [29, 31]:

$$R\left(T, \frac{\partial T}{\partial z}, \frac{\partial T}{\partial t}\right) = \left|\alpha T + \beta \frac{\partial T}{\partial z} + \gamma \frac{\partial T}{\partial t}\right| \quad (12)$$

or

$$R\left(T, \frac{\partial T}{\partial t}\right) = \left|\alpha T + \gamma \frac{\partial T}{\partial t}\right|. \quad (13)$$

By eliminating the time derivative with the aid of equation (1), these boundary conditions are reduced to a non-classical boundary condition containing the second derivatives

$$R\left(t, \frac{\partial T}{\partial z}, \frac{\partial T}{\partial t}\right) = \left|\alpha T + \beta \frac{\partial T}{\partial z}\right| + \gamma\left(\frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial z^2} - \frac{\kappa}{\rho c_p}T + \frac{F}{\rho c_p}\right). \quad (14)$$

The boundary conditions involving higher derivatives were introduced into the heat conduction theory by Samarsky [31] and Tikhonov [32]. Tikhonov also considered the case when the order of the derivatives in the boundary condition can exceed the order of the equation itself

$$\sum_{k=0}^{m} \alpha_k \frac{\partial^k T}{\partial z^k} = \Psi_k. \quad (15)$$

The boundary conditions containing the space and time derivatives, as well as higher and mixed derivatives, also occur naturally in the theory of heat and mass transfer in the case of finite rate of heat and mass propagation which is based on the generalized laws of Fourier [1, 33, 40]

$$\mathbf{q} = -\lambda \operatorname{grad} T - \tau_q \frac{\partial \mathbf{q}}{\partial t} \quad (16)$$

and of Fick

$$\mathbf{j}_m = -D \operatorname{grad} c - \tau_m \frac{\partial \mathbf{j}_m}{\partial t} \quad (17)$$

with regard for relaxation processes at the finite relaxation times τ_q and τ_m [42].

The heat balance equation

$$\rho c_p \frac{\partial T}{\partial t} = -\operatorname{div} \mathbf{q} - \kappa T + F \quad (18)$$

yields, instead of equation (1), the hyperbolic heat conduction equation, for example, for temperatures

$$\tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \rho c_p \frac{\partial T}{\partial t} = \lambda \Delta T - \tau_q \kappa \frac{\partial T}{\partial t} - \kappa T + \tau_q \frac{\partial F}{\partial t} + F \quad (19)$$

and a similar equation for the flux $\mathbf{q}$.

Equation (17) and mass conservation law (continuity equation)

$$\frac{\partial c}{\partial t} = -\operatorname{div} \mathbf{j}_m - Kc + S \quad (20)$$

yield the hyperbolic equation of mass transfer with the finite rate of mass propagation

$$\tau_m \frac{\partial^2 c}{\partial t^2} + \frac{\partial c}{\partial t} = D\Delta c - \tau_m K \frac{\partial c}{\partial z} - Kc + \tau_m \frac{\partial S}{\partial t} + S \quad (21)$$

which is analogous in structure to the hyperbolic heat conduction equation (19). The systems of generalized heat and mass transfer equations (16)–(18), (20) are analogous to the hyperbolic equations of long lines

$$\begin{aligned} \frac{\partial u}{\partial z} &= -L\frac{\partial i}{\partial t} - Ri \\ \frac{\partial i}{\partial z} &= -C\frac{\partial u}{\partial t} - Gu \end{aligned} \quad (22)$$

or to the Maxwell equations [41]

$$\begin{aligned} \frac{\partial E}{\partial z} &= -\mu \frac{\partial H}{\partial t} \\ \frac{\partial H}{\partial z} &= -\varepsilon \frac{\partial E}{\partial t} + \sigma E. \end{aligned} \quad (23)$$

This analogy was already noted in 1968 [33–41] and was widely employed to develop the nonlinear transfer theory [29]. Natural boundary conditions (i)–(v) also take place for the generalized hyperbolic heat conduction equation, but then they are transformed into generalized boundary conditions with first derivatives of the form of conditions (12), (13) or into those with higher or mixed derivatives.

Obtain for u the second-order equation and similar equations for i, E, H. Take, for example, the fourth kind boundary condition (6)

$$\frac{\partial^2 u}{\partial z^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC+GL)\frac{\partial u}{\partial t} + GRu. \quad (24)$$

To solve equation (19), it is necessary to eliminate the flux q from equation (24) with the aid of equation (16)

$$T_1|_{z=a+0} = T_2|_{z=a-0}, \left|\lambda_1 \frac{\partial T_1}{\partial z} + \tau_{q_1}\frac{\partial q_1}{\partial t}\right|_{z=a+0} = \left|\lambda_2 \frac{\partial T_1}{\partial z} + \tau_{q_2}\frac{\partial q_2}{\partial t}\right|_{z=a-0}. \quad (25)$$

Differentiating this equation with respect to z and equation (18) with respect to t and eliminating $(\partial^2 q)/(\partial z\, \partial t)$, the following boundary condition will be obtained

$$\left|\lambda_1 \frac{\partial^2 T_1}{\partial z^2} + \tau_{q_1}\left(\frac{\partial F_1}{\partial t} - \rho_1 c_{p_1}\frac{\partial^2 T_1}{\partial t^2} - \kappa_1 \frac{\partial T_1}{\partial t}\right)\right|_{z=a+0} = \left|\lambda_2 \frac{\partial^2 T_2}{\partial z^2} + \tau_{q_2}\left(\frac{\partial F_2}{\partial t} - \rho_2 c_{p_2}\frac{\partial^2 T_2}{\partial t^2} - \kappa_2 \frac{\partial T_2}{\partial t}\right)\right|_{z=a-0} \quad (26)$$

from which it is seen that boundary condition (26) involves the second space and time derivatives of the unknown function, the first time derivatives of the unknown function T and of the known source F. These and more general boundary conditions of any other order were formulated by the author in his works [29, 42–44].

The boundary conditions involving higher derivatives were considered by Ostrogradsky [7] and Krylov [45] in the theory of oscillations of strings and rods loaded at the ends by concentrated masses; for example, the equation of string oscillations [45] was considered

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial z^2} + F(t, z) \quad (27)$$

with the conditions on one end

$$\frac{\partial^2 v}{\partial t^2} = -c^2 \frac{\partial v}{\partial z} \quad (28)$$

which leads to the boundary condition

$$a^2 \frac{\partial^2 v}{\partial z^2} + c^2 \frac{\partial v}{\partial z} = F(z, t) \quad (29)$$

involving the first and second spatial derivatives or the second time derivative and the first space derivative.

In the theory of wave motions of an ideal fluid with a free surface, Ostrogradsky obtained for the potential the following boundary condition on the free surface in the linear approximation [7]:

$$\left|\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z}\right|_{z=a} = 0 \quad (30)$$

which also contains the second time derivative and the first space derivative.

In the theory of long lines one encounters the boundary conditions of the form [46]:

$$R_0 \frac{\partial i}{\partial t} + L_0 \frac{\partial^2 i}{\partial t^2} + \frac{1}{C_0} i = \frac{\partial u}{\partial t} + \frac{\partial E}{\partial t} \quad (31)$$

that contain the first and second time derivatives of the function, the function proper and the derivative of the source.

The boundary-value problems with higher derivatives were considered in the elasticity theory, in the theory of boundary-value problems with higher derivatives [10]. This allows one to isolate the boundary conditions containing higher and mixed derivatives into a special class, which are called the boundary conditions of the sixth kind (vi) or Ostrogradsky–Samarsky–Tikhonov's conditions [28, 29].

And, finally, the generalized statement of boundary linear conditions for linear equations can be presented in the form of a linear operator of boundary conditions involving in a general case the derivatives of any order [42]

$$\Phi = R\left(T_1, \frac{\partial T}{\partial t}, \frac{\partial T}{\partial z}, \frac{\partial^2 T}{\partial t^2}, \frac{\partial^2 T}{\partial z^2}, \frac{\partial^2 T}{\partial z\, \partial t}, \ldots\right) \quad (32)$$

where R is the linear differential operator of the nth order. Restricting the discussion, for example, to the second derivatives, this operator can be presented in a general form as follows [49]:

$$\left|\alpha_{1i}\frac{\partial^2 T_i}{\partial z^2} + \alpha_{2i}\frac{\partial^2 T_i}{\partial z\, \partial t} + \alpha_{3i}\frac{\partial^2 T_i}{\partial t^2} + \alpha_{4i}\frac{\partial T_i}{\partial z} + a_{5i}\frac{\partial T_i}{\partial t} + \alpha_{6i} T\right|_{\Gamma+0} + \left|\beta_{1j}\frac{\partial^2 T_j}{\partial z^2} + \beta_{2j}\frac{\partial^2 T_j}{\partial z\, \partial t} + \beta_{3j}\frac{\partial^2 T_j}{\partial t^2} + \beta_{4j}\frac{\partial T_j}{\partial z} + \beta_{5j}\frac{\partial T_j}{\partial t} + \beta_{6j} T\right|_{\Gamma-0} = \Psi_1 \quad (33)$$

where α_{ij} and β_{ij} are the coefficients, Ψ_i is the function, or it can be reduced to different canonical forms. These boundary conditions with the second-order derivatives include all previous boundary conditions (i)–(vi) with the derivatives up to the second-order, inclusive. The boundary condition of the general form (32) formally includes all possible differential-type linear conditions. Also possible is the statement of generalized non-linear differential-type boundary conditions [29].

GENERALIZED BOUNDARY CONDITIONS OF INTEGRAL AND INTEGRODIFFERENTIAL TYPES

The boundary conditions of the transport theory can be formulated in integral and integrodifferential forms. The different forms of integral-type boundary conditions for parabolic equations were considered in refs. [2, 47, 48].

The solution of equation (16) with the initial condition

$$t = 0, \quad T = T_0(z), \quad q = q_0(z) \quad (34)$$

will be written in the form

$$q(z,t) = -\int_0^t \frac{\lambda}{\tau_q}\frac{\partial T}{\partial z} e^{-(\tau - t)/\tau_q} + q_0(z)\, e^{-(t/\tau_q)}. \quad (35)$$

Any boundary condition containing the heat flux will acquire, when converting to the description in terms of temperature, the form of an integrodifferential equation. For example, conditions (25) will take the form

$$T_1|_{z=a+0} = T_2|_{z=a-0}$$

$$\left| -\int_0^t \frac{\lambda_1}{\tau_{q_1}}\frac{\partial T_1}{\partial z} e^{-(\tau-t)/\tau_{q_1}}\,\mathrm{d}\tau + q_0(z)\, e^{-(t/\tau_{q_1})} \right|_{z=a+0}$$

$$= \left| -\int_0^t \frac{\lambda_2}{\tau_{q_2}}\frac{\partial T_2}{\partial z} e^{-(\tau-t)/\tau_{q_2}}\,\mathrm{d}\tau + q_0(z)\, e^{-(t/\tau_{q_2})} \right|_{z=a-0}. \quad (36)$$

Boundary conditions (ii) and (iii), that contain the heat flux, can be transformed in a similar fashion. By substituting equation (35) into equation (18) the following integrodifferential equation can be obtained

$$\rho c_p \frac{\partial T}{\partial t} = \int_0^t \frac{\lambda}{\tau_q}\frac{\partial^2 T}{\partial z^2} e^{-(\tau-t)/\tau_q}\,\mathrm{d}\tau + \frac{\partial}{\partial z} q_0(z)\, e^{-(t/\tau_q)} - \kappa T + F. \quad (37)$$

The boundary condition (31) for the theory of long lines is written down in integrodifferential forms as

$$L_0 \frac{\partial i}{\partial t} + R_0 i + \frac{1}{C_0}\int_0^t i\,\mathrm{d}t = u + E. \quad (38)$$

The mating of electrical circuits can be done via different elements (capacitor, inductor, resistor, etc.) and this gives rise to a much greater diversity of the types of conjugate problems.

The integration of the second equation of system (32) over t yields

$$u = -\int_0^t \frac{1}{C}\frac{\partial i}{\partial z} e^{-[(\tau-t)G/C]} + u_0(z)\, e^{-(tG/C)} \quad (39)$$

whence it is seen that the relaxation time in equation (32) is determined by the expression $\tau_p = C/G$. Then, to determine i from equation (38), the following integrodifferential equation is obtained

$$L_0 \frac{\partial i}{\partial t} + R_0 i = \frac{1}{C_0}\int_0^t \frac{\partial^2 i}{\partial z^2} e^{-[(\tau-t)G/C]} - \frac{\partial u_0}{\partial z} e^{-(tG/C)} \quad (40)$$

while equation (31) yields the following integrodifferential boundary condition

$$L_0 \frac{i}{\mathrm{d}t} + R_0 i + \frac{1}{C_0}\int_0^t i\,\mathrm{d}t = E - \frac{1}{C_0}\int_0^t \frac{\partial i}{\partial z} e^{-[(\tau-t)G/C]}\,\mathrm{d}\tau + u_0\, e^{-(tG/C)}. \quad (41)$$

Similarly one can obtain a number of other boundary conditions of integral or integrodifferential forms [42].

On returning to the equations and boundary conditions of intensive heat and mass transfer it can be seen that equation (19) describes the propagation of heat with the finite rate and, at the same time, as an integrodifferential equation, it takes into account the prehistory of the process. The study of integrodifferential equations for media with memory was started long ago in the hereditary elasticity theory [37, 48], where the dynamic equations of the medium with memory result from ordinary equations by replacing the material characteristics of the medium by the integral operator of heredity. By taking into account the ordinary and hereditary operators of the heat flux in the form

$$q^* = -\lambda \nabla T - \int_0^t \frac{\lambda'}{\tau_q} \nabla T K(t-\tau)\,\mathrm{d}\tau \quad (42)$$

and the operators of the ordinary and hereditary mass density of the thermal energy

$$(\rho c_p T)^* = \gamma(0) T + \int_0^t \gamma' T\,\mathrm{d}\tau \quad (43)$$

the equations of the form of (18), (42) and (43) will give the following integrodifferential heat conduction equation for media with memory [50]

$$\gamma \frac{\partial T}{\partial t} + \int_0^t \gamma' \frac{\partial T}{\partial t}\,\mathrm{d}\tau = \lambda \frac{\partial^2 T}{\partial z^2} + \int_0^t \lambda' \frac{\partial^2 T}{\partial z^2} K(t-\tau)\,\mathrm{d}\tau - \kappa T + F \quad (44)$$

where $\gamma(0)$ is the instantaneous volumetric heat capacity and $\lambda(0)$ are the instantaneous thermal conductivity coefficients, γ' is the relaxation function of the internal energy, $K(t-\tau)$ is the integral equation kernel. The initial conditions at $t = 0$ are usually prescribed in the form

$$\left| \gamma(0) T + \int_0^t \alpha' T\,\mathrm{d}\tau \right|_{t=t_0} = T_0(z) \quad (45)$$

$$\left| -\lambda \frac{\partial T}{\partial z} - \frac{\lambda'}{\tau_q}\int_0^t K'(t-\tau)\frac{\partial T}{\partial z}\,\mathrm{d}\tau \right| = q_0(z). \quad (46)$$

The boundary conditions for media with relaxation or memory result from the substitution of the integral operators of heredity into corresponding boundary conditions for the expressions for ordinary media. Thus, for example, equations (5), (42) and (43) yield the boundary conditions of integrodifferential type

$$\left| \alpha T + \alpha \int_0^t \alpha' T\,\mathrm{d}\tau - \beta\lambda \nabla T - \beta\lambda' \int_0^t K(t-\tau)\nabla T\,\mathrm{d}\tau \right| = \xi. \quad (47)$$

Other integrodifferential boundary conditions are obtained in a similar manner.

The mathematical aspects of the generalized boundary conditions are discussed in refs. [50–60].

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CONDITIONS AUX LIMITES GENERALISEES DU TRANSFERT DE CHALEUR ET DE MASSE

Résumé—On considère la genèse de l'établissement des conditions aux limites classiques et généralisées en relation avec les équations paraboliques et hyperboliques de la chaleur. Des conditions aux limites de type différentiel d'ordre quelconque qui impliquent les conditions aux limites classiques sont généralisées pour les équations hyperboliques ; on montre la possibilité de formuler les conditions aux limites sous des formes intégrales ou intégro-différentielles pour les équations hyperboliques dans la théorie du transfert de chaleur et de masse, de l'électro-dynamique et de la thermodynamique des milieux avec mémoire ou avec des mécanismes de relaxation.

VERALLGEMEINERTE RANDBEDINGUNGEN IN DER WÄRME- UND STOFFÜBERTRAGUNG

Zusammenfassung—Es wird die Formulierung klassischer und verallgemeinerter Randbedingungen für parabolische und hyperbolische Wärmeleitgleichungen betrachtet. Randbedingungen jeglicher Ordnung vom Differential-Typ, die alle bekannten früheren Randbedingungen beinhalten, wurden für hyperbolische Gleichungen verallgemeinert. Es wird die Möglichkeit gezeigt, Randbedingungen in Integral- oder Integrodifferential-Form für hyperbolische Gleichungen in der Wärme- und Stoffübertragungstheorie der Elektrodynamik und der Thermomechanik von Stoffen mit Gedächtnis oder mit Relaxation, zu formulieren.

ОБОБЩЕННЫЕ ГРАНИЧНЫЕ УСЛОВИЯ ТЕОРИИ ТЕПЛО-И МАССОПЕРЕНОСА

Аннотация—Рассмотрен генезис постановки классических и обобщенных краевых условий на примере параболического и гиперболического уравнения теплопроводности. Обобщены граничные условия дифференциального типа любых порядков для гиперболических уравнений, содержащие в себе все известные ранее граничные условия, показана возможность формулировки краевых условий в интегральном или интегродифференциальном виде для гиперболических уравнений в теории тепло-и массопереноса, электродинамике и термомеханике сред с памятью или релаксационными процессами.